SOME RESULTS ON FIXED POINT OF FUNCTION IN S-METRIC SPACES

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Abstract

In this paper, we extend a version of Caristi's fixed point theorem proved by Bollenbacher and Hicks [*Proc. Amer. Math. Soc. 1988; 102(4): 898-900*] *to* S *-metric spaces. We also derive some fixed point theorems from our main result.*

Keywords: Caristi's fixed point theorem, S -metric spaces.

INTRODUCTION AND PRELIMINARIES

In 1988 Bollenbacher and Hicks [1] proved a version of famous Caristi's fixed point theorem [2]. Bollenbacher and Hicks showed in [1] that " *Let* (X,d) *be a metric space. Suppose* $T: X \to X$ *and* $\phi : X \to [0,\infty)$. *Suppose there exists an* x *such that*

$$d(y,Ty) \le \phi(y) - \phi(Ty)$$

for every $y \in O(x,\infty)$, *and any Cauchy sequence in* $O(x,\infty)$ *converges to a point in* X . *Then:*

(1) $\lim T^n x = \bar{x}$ *exists.*
(2) $d(T^n x, \bar{x}) \le \phi(T^n x)$.
(3) $T\bar{x} = \bar{x}$ *iff* $G(x) = d(x,Tx)$ *is* T *-orbitally lower semicontinuous at* x .
(4) $d(T^n x, x) \le \phi(x)$ *and* $d(\bar{x}, x) \le \phi(x)$. "

In this theorem saying that for $x \in X$, $O(x,\infty) = \{x, Tx, T^2x, ...\}$ is the orbit of x .

Recently, Sedghi et al. [5] have introduced the concept of S -metric spaces and give a fixed point theorem for self-mapping on complete S -metric spaces.

In this paper, we extend the result of Bollenbacher and Hick's to S -metric spaces.

We now recall some definitions and properties for S -metric spaces by Sedghi at al. [5].

Definition 1.1 [5] Let X be a nonempty set. A function $S : X^3 \to [0,\infty)$ is said to be an S -metric on X , if for each $x, y, z, a \in X$,

(1) $S(x, y, z) \ge 0$,
(2) $S(x, y, z) = 0$ if and only if $x = y = z$,
(3) $S(x, y, z) \le S(x, x, a) + S(y, y, a) + S(z, z, a)$

The pair (X,S) is called an S -metric space.

Example 1.2 [5]

(1) Let $X = IR^n$ and $\| . \|$ a norm on X .Then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S -metric on X .
(2) Let $X = IR^n$ and $\| . \|$ a norm on X . Then

$S(x,y,z)=\|x-z\|+\|y-z\|$ is an S-metric on X.

(3) Let X be a nonempty set and d be an ordinary metric on X. Then $S(x,y,z)=d(x,z)+d(y,z)$ is an S-metric on X.

Lemma 1.3 [5] *Let (X,S) be an S-metric space. Then, we have $S(x,x,y)=S(y,y,x)$ for all $x,y\in X$.*

Definition 1.4 [5] Let (X,S) be an S-metric space and $A\subset X$.

(1) A sequence $\{x_n\}$ in X converges to x if $S(x_n,x_n,x)\to 0$ as $n\to\infty$, that is for every $\varepsilon>0$ there exists $n_0\in IN$ such that for $n\geq n_0$, $S(x_n,x_n,x)<\varepsilon$. In this case, we denote by $\lim_{n\to\infty}x_n=x$ and we say that x is limit of $\{x_n\}$ in X.

(2) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for each $\varepsilon>0$,there exists $n_0\in IN$ such that $S(x_n,x_n,x_m)<\varepsilon$ for each $n,m\geq n_0$.

(3) The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.

Lemma 1.5 [5] *The limit of $\{x_n\}$ in S-metric space (X,S) is unique.*

Lemma 1.6 [5] *Let (X,S) be an S-metric space. Then the convergent sequence $\{x_n\}$ in X is Cauchy.*

Lemma 1.7 [5] *Let (X,S) be an S-metric space. If there exist sequence $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}y_n=y$, then $\lim_{n\to\infty}S(x_n,x_n,y_n)=S(x,x,y)$.*

Definition1.8 Let (X,S) be an S-metric space and $T:X\to X$ a mapping of X. The set $O(x,\infty)=\{x,Tx,T^2x,\ldots\}$ is called the orbit of X.

If for an $x\in X$, every Cauchy sequence in $O(x,\infty)$ converges to a point in X, then the S-metric space is said to be (x,T)-orbitally complete.

Definition 1.9 Let (X,S) be an S-metric space and $T:X\to X$ a mapping of X. A real-valued function $F:X\to[0,\infty)$ is said to be T-*orbitally weak semi continuous* (w.l.s.c.) *at* p *relative to* x iff $\{x_n\}$ is a sequence in $O(x,\infty)$ and $\lim_{n\to\infty}x_n=p$ implies $F(p)\leq\limsup_{n\to\infty}F(x_n)$.

Clearly, every function F that is T-orbitally lower semi continuous (l.s.c.) at p relative to $x\in X$ (that is, $\{x_n\}\subseteq O(x,\infty)$ and $\lim x_n=p$ implies $F(p)\leq\liminf_{n\to\infty}F(x_n)$, see [1] is also T-orbitally w.l.s.c. at p relative to x, but the implications is not reversible, see [3].

MAIN RESULTS

Several authors have obtained various example [4,6,7], and others.

We now extend the results of Bollenbacher and Hicks [1] to S-metric spaces.

Theorem 2.1. Let (X,S) be an S-metric space, $T:X\to X$ *and* $\psi:X\to[0,\infty)$. Suppose there exists an $x\in X$ such that

$$S(y,y,Ty)\leq\psi(y)-\psi(Ty) \qquad (1)$$

for all $y\in O(x,\infty)$, *and* (x,T)-orbitally complete. Then:

(a) $\lim T^n x=x'\in X$ *exists,*

(b) $S(T^n x,T^n x,x')\leq 2\psi(T^n x)$,

(c) $Tx'=x'$ *if and only if* $F(z)=S(x,x,Tx)$ is T-*orbitally* w.l.s.c. at x' relative x.

Proof: (a) Using inequality (1) we have

$$S_n = \sum_{k=0}^{n} S\left(T^k x, T^k x, T^{k+1}x\right)$$
$$\le \sum_{k=0}^{n}\left[\psi\left(T^k x\right) - \psi\left(T^{k+1}x\right)\right]$$
$$= \psi(x) - \psi\left(T^{n+1}x\right) \le \psi(x)$$

Therefore $\{S_n\}$ is bounded above and also non-decreasing and also convergent.

Let $m > n$ then from property (3) of S-metric and Lemma 1.3 , we have

$$S\left(T^n x, T^n x, T^m x\right)$$
$$\le 2\,S\left(T^n x, T^n x, T^{n+1}x\right) + S\left(T^m x, T^m x, T^{n+1}x\right)$$
$$= 2\,S\left(T^n x, T^n x, T^{n+1}x\right) + S\left(T^{n+1} x, T^{n+1} x, T^{m}x\right)$$
$$\le 2\left[S\left(T^n x, T^n x, T^{n+1}x\right) + S\left(T^{n+1} x, T^{n+1} x, T^{n+2}x\right)\right] + S\left(T^m x, T^m x, T^{n+2}x\right)$$
$$= 2\left[S\left(T^n x, T^n x, T^{n+1}x\right) + S\left(T^{n+1} x, T^{n+1} x, T^{n+2}x\right)\right] + S\left(T^{n+2} x, T^{n+2} x, T^{m}x\right)$$
$$\vdots$$
$$\le 2\sum_{k=n}^{m-2} S\left(T^k x, T^k x, T^{k+1}x\right) + S\left(T^{m-1} x, T^{m-1} x, T^{m}x\right)$$
$$\le 2\sum_{k=n}^{m-1} S\left(T^k x, T^k x, T^{k+1}x\right)$$

(2) Since $\{S_n\}$ is convergent, for every $\varepsilon > 0$ we can choose a sufficiently large $N \in IN$ such that

$$\sum_{k=n}^{\infty} S\left(T^k x, T^k x, T^{k+1}x\right) < \frac{\varepsilon}{2}$$

for all $n > N$. Thus we get from (2) that

$$S\left(T^n x, T^n x, T^m x\right) < \varepsilon$$

for all $m, n \ge N$, and so $\{T^n x\}$ is a Cauchy sequence in $O(x,\infty)$. Since (X,S) is (x,T)-orbitally complete, $\lim T^n x = x'$ exists.

(b) Using (1) and (2) we have

$$S\left(T^n x, T^n x, T^m x\right) \le 2\sum_{k=n}^{m-1} S\left(T^k x, T^k x, T^{k+1}x\right)$$
$$\le 2\sum_{k=n}^{m-1}\left[\psi\left(T^k x\right) - \psi\left(T^{k+1}x\right)\right]$$
$$= 2\left[\psi\left(T^n x\right) - \psi\left(T^m x\right)\right]$$
$$\le 2\psi\left(T^n x\right)$$

Letting m tend to infinity, we have from (a) and Lemma 1.7.

$$S\left(T^n x, T^n x, x'\right) \le 2\psi\left(T^n x\right).$$

(c) Assume that $Tx' = x'$ and $\{x_n\}$ is a sequence in $O(x,\infty)$ with $\lim x_n = x'$. Then

$$F(x') = S(x', x', Tx') \le \limsup S(x_n, x_n, Tx_n)$$
$$= \limsup F(x_n),$$

and so F is T-*orbitally* w.l.s.c. at x' relative x.

Now let $x_n = T^n x$ and F is T-*orbitally* w.l.s.c. at x' relative x. Then from (a) and Lemma 1.7. we have

$$0 \le S(x', x', Tx') = F(x') \le \limsup F(x_n)$$
$$= \limsup S\left(T^n x, T^n x, T^{n+1}x\right) = 0$$

Thus $Tx' = x'$.

Definition 2.2. [5] Let (X,S) be an S-metric space. A map $F : X \to X$ is said to be a contraction if there exists a constant $0 \le L < 1$ such that

$$S(F(x), F(x), F(y)) \le L\,S(x, x, y),$$

for all $x, y \in X$.

From Theorem 2.1 we obtain the following corollary which is slight generalization of [5].

Corollary 2.3. Let (X,S) be an S-metric space and T be a self mapping of X. Suppose there exists an $x \in X$ such that T be a contraction mapping for all $y \in O(x,\infty)$, and (X,S) is (x,T)-orbitally complete then $\lim T^n x = x' \in X$ exists and x' is a unique fixed point of T. Furthermore,

$$S\left(T^n x, T^n x, x'\right) \le \frac{2L^n}{1-L} S(x, x, Tx)$$

where L is a contraction constant.

Proof: Define $\psi(y) = \frac{1}{1-L} S(y, y, Ty)$ for all $y \in O(x,\infty)$. Since T is a contraction, we have

$$\psi(Ty) = \frac{1}{1-L} S(Ty, Ty, T^2 y)$$
$$\leq \frac{L}{1-L} S(y, y, Ty)$$
$$= L\psi(y).$$

Thus we get,

$$S(y, y, Ty) = (1-L)\psi(y)$$
$$= \psi(y) - L\psi(y)$$
$$\leq \psi(y) - \psi(Ty).$$

Then $\lim T^n x = x' \in X$ follow immediately from Theorem 2.1.

Since T is a contraction, T is a continuous mapping and so from Lemma 1.7 we get F is T-*orbitally* w.l.s.c. at x' relative x. Thus $Tx' = x'$ from Theorem 2.1. Using (b) of Theorem 2.1 we have,

$$S(T^n x, T^n x, x') \leq 2\psi(T^n x)$$
$$\leq 2L\, \psi(T^{n-1} x)$$
$$\leq 2L^2\, \psi(T^{n-2} x)$$
$$\vdots$$
$$\leq 2L^n\, \psi(x)$$
$$= 2\frac{L^n}{1-L} S(x, x, Tx).$$

Corollary 2.4: Let (X, S) be an S-metric space and T be a self mapping of X. Suppose there exists an $x \in X$ such that,

$$S(Ty, Ty, T^2 y) \leq L\, S(y, y, Ty) \qquad (3)$$

for all $y \in O(x, \infty)$ where $0 < L < 1$, and (X, S) is (x, T)-orbitally complete. Then

(a) $\lim T^n x = x'$ exists,

(b) $S(T^n x, T^n x, x') \leq \frac{2L^n}{1-L} S(x, x, Tx)$,

(c) $Tx' = x'$ *if and only if* $F(z) = S(x, x, Tx)$ is T-*orbitally* w.l.s.c. at x' relative x.

Proof: Put $\psi(y) = \frac{1}{1-L} S(y, y, Ty)$ for all $y \in O(x, \infty)$. Let $y \in T^n x$ in (3), then we have

$$S(T^{n+1} x, T^{n+1} x, T^{n+2} x) \leq L\, S(T^n x, T^n x, T^{n+1} x)$$

and

$$S(T^n x, T^n x, T^{n+1} x) - L\, S(T^n x, T^n x, T^{n+1} x)$$
$$\leq S(T^n x, T^n x, T^{n+1} x) - S(T^{n+1} x, T^{n+1} x, T^{n+2} x)$$

and so

$$S(T^n x, T^n x, T^{n+1} x) \leq \frac{1}{1-L} \left[S(T^n x, T^n x, T^{n+1} x) - S(T^{n+1} x, T^{n+1} x, T^{n+2} x)\right]$$

Thus we get

$$S(y, y, Ty) \leq \psi(y) - \psi(Ty)$$

so (a) and (c) are immediate from Theorem 2.1. Using inequality (3) we have

$$S(T^n x, T^n x, T^{n+1} x) \leq L^n\, S(x, x, Tx)$$

and then from Theorem 2.1 (b) we get

$$S(T^n x, T^n x, x') \leq 2\psi(T^n x)$$
$$= 2\frac{1}{1-L} S(T^n x, T^n x, T^{n+1} x)$$
$$\leq 2\frac{L^n}{1-L} S(x, x, Tx)$$

and this gives (b).

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